

# Analytical mechanics of a relativistic particle in a positional potential

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## Abstract

We propose a form for the action of a relativistic particle subject to a positional force that is invariant under time reparametrization and therefore allows for a consistent Hamiltonian formulation of the dynamics. This approach can be useful in the study of phenomenological models. Also the Dirac and Klein-Gordon equation differ from the standard formulation, with corrections of order  $(E - m)/m$  in the energy spectra.

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## 1. Introduction

Analytical mechanics is at the basis of modern theoretical physics. In particular, Hamiltonian methods are largely employed in several applications and are essential for the formulation of the quantum theory. However, not all systems have been consistently described in Hamiltonian terms. For example, if one excludes fundamental interaction, like Maxwell theory, whose coupling to particles is dictated by gauge invariance, in special relativity it is not easy to introduce particle interactions that preserve the Lorentz and the reparametrization invariance of the action. While for phenomenological interactions the request of Lorentz invariance is not compelling, the breakdown of the reparametrization invariance has serious consequences on the possibility of defining a consistent Hamiltonian for the model under study. The lack of a Hamiltonian formulation also implies that in quantum theory the Klein-Gordon or Dirac equations for generic interactions have to be postulated rather than derived from the correspondence principle.

These problems arise especially for models like that of an external central force acting on a particle, when the force is fixed a priori and is not determined by field equations derived from a variational principle. An important case is that of the harmonic oscillator, whose relativistic formulation is problematic [1]. Usually, in such cases the Hamiltonian is defined only for a specific choice of the time coordinate.

In this paper, we propose a different solution that, slightly modifying the coupling, makes the action reparametrization invariant. This allows to obtain a Hamiltonian by means of the usual Dirac formalism for constrained systems. The corrections with respect to the standard formalism are of order  $(E - m)/m$ , where  $E$  is the relativistic energy and  $m$  the mass of the particle. Moreover, a really covariant Klein-Gordon or Dirac equation can be obtained. In the case of the harmonic oscillator, a proposal leading to equivalent results was advanced long ago in [1], starting from different considerations.

The paper is organized as follows: in sect. 2 we review the Lagrangian and Hamiltonian formalism for a free particle. In sect. 3 we describe our proposal and compare it with the standard formulation of the relativistic interacting particle. In sect. 4 we give some elementary instances of application of the formalism. In sect. 5 the changes in the Klein-Gordon equations are discussed and illustrated with simple examples in sect. 6.

## 2. Free particle

In special relativity, the motion of a free particle in coordinates  $x^\mu$  is determined by the variation of the action

$$S = \int \mathcal{L} d\tau = -m \int \sqrt{(\dot{x}^\rho)^2} d\tau = -m \int ds, \quad (1)$$

where  $\mathcal{L}$  is the Lagrangian density,  $\tau$  an arbitrary evolution parameter, a dot denotes a derivative with respect to  $\tau$ ,  $m$  is the rest mass and  $s$  the proper time on the trajectory. The action is invariant under Lorentz transformations and under reparametrizations  $\tau \rightarrow \tau'(\tau)$ . The Euler-Lagrange equations for the action (1) read

$$m \frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} = 0. \quad (2)$$

The dynamics can also be written in Hamiltonian form [2]. The momentum conjugated to  $x^\mu$  is defined as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m \frac{\dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} = m \frac{dx^\mu}{ds}, \quad (3)$$

and satisfies the constraint

$$p_\mu^2 = m^2. \quad (4)$$

Since the action is reparametrization invariant, the Hamiltonian  $\mathcal{H}$  vanishes, and the action can be written in first-order form as

$$S = \int d\tau (\dot{x}^\mu p_\mu - \lambda \mathcal{H}), \quad \mathcal{H} = p_\mu^2 - m^2, \quad (5)$$

where  $\lambda$  is a Lagrange multiplier enforcing the constraint (4).

The Hamilton equations ensuing from the action (5) are

$$\dot{x}_\mu = \{x_\mu, \lambda \mathcal{H}\} = 2\lambda p_\mu, \quad \dot{p}_\mu = \{p_\mu, \lambda \mathcal{H}\} = 0, \quad (6)$$

from which, comparing with (3), follows that

$$\lambda = \frac{\sqrt{(\dot{x}^\rho)^2}}{2m}. \quad (7)$$

In Dirac formalism, the constraint (4) is first order. One can therefore reduce the system by choosing a gauge, i.e. fixing the time coordinate.

The standard choice is  $t = x^0$ , namely one identifies the evolution time  $t$  with the coordinate time  $x_0$ . Since  $\{t, \lambda \mathcal{H}\} = 2\lambda p_0$ , no secondary constraints arise and one easily checks that

$$x'_i = \frac{p_i}{p_0} = \frac{p_i}{\sqrt{p_k^2 + m^2}}, \quad p'_i = 0, \quad \lambda = \frac{1}{2m} \sqrt{1 - x_k'^2} = \frac{1}{2p_0} \quad (8)$$

where  $i = 1, 2, 3$  and a prime denotes a derivative with respect to  $t$ . The reduced action is then

$$S = \int dt \left( x'_i p_i - \sqrt{p_i^2 + m^2} \right), \quad (9)$$

and the 3-dimensional effective Hamiltonian  $H = p_0 = \sqrt{p_i^2 + m^2}$  can be identified with the energy of the particle in the laboratory frame.

Other choices of the gauge are however possible. For example, with the choice  $t = x^\mu p_\mu$ , the evolution time coincides with the proper time of the particle.

### 3. Particle in an external potential

The addition of an external potential acting on the free particle presents some problems and to our knowledge has not been discussed in depth. In the standard formalism, one simply adds to the action a potential term

$$S_{int} = - \int V(x, \dot{x}) d\tau. \quad (10)$$

In general the potential breaks the Lorentz invariance. Moreover, unless  $V$  is homogeneous of degree one in the velocity, as in the Maxwell case when  $V = \dot{x}^\mu A_\mu(x)$ ,  $S_{int}$  is not invariant under reparametrization, and the Dirac formalism cannot be consistently applied for writing down the equations of motion in Hamiltonian form.

In the following we discuss the case of positional potentials  $V = V(x)$  that do not depend on the velocity of the particle. These usually arise as phenomenological potentials, like those associated to elastic forces. For positional potentials, the interaction term (10) leads to the standard equations of motion

$$\frac{d}{d\tau} \frac{m \dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} - \partial^\mu V = 0. \quad (11)$$

Multiplying eq. (11) by  $\dot{x}_\mu$  and integrating, one can show that the total energy

$$E = \frac{m}{\sqrt{(\dot{x}^\rho)^2}} + V \quad (12)$$

is conserved. Since  $V = V(x)$ , the momenta are given by (3) as for the free particle and satisfy the constraint (4). If one tries to apply the Dirac formalism to the action so defined, one obtains that  $V$  must be a constant. This is a consequence of the lack of reparametrization invariance of (10).

In order to avoid these problems and to preserve reparametrization invariance, we propose an interaction term of the form

$$S_{int} = - \int \sqrt{(\dot{x}^\rho)^2} V(x) d\tau, \quad (13)$$

leading to the total action

$$S = - \int \sqrt{(\dot{x}^\rho)^2} (m + V) d\tau. \quad (14)$$

The Euler-Lagrange equations for the action (14) read then

$$\frac{d}{d\tau} \frac{(m + V) \dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} - \sqrt{(\dot{x}^\rho)^2} \partial^\mu V = 0. \quad (15)$$

If  $V$  does not depend on  $\dot{x}^\mu$ , eq. (15) can also be written

$$(m + V) \frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} + \frac{\dot{x}^\mu \dot{x}^\nu \partial_\nu V - (\dot{x}^\nu)^2 \partial^\mu V}{\sqrt{(\dot{x}^\rho)^2}} = 0. \quad (16)$$

One can again find the energy integral by multiplying eq. (15) by  $\dot{x}_\mu$  and integrating, obtaining

$$E = \frac{m + V}{\sqrt{(\dot{x}^\rho)^2}}. \quad (17)$$

To pass to the Hamiltonian formulation, we compute the momentum conjugate to  $x^\mu$ ,

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = (m + V) \frac{\dot{x}^\mu}{\sqrt{(\dot{x}^\rho)^2}} = (m + V) \frac{dx^\mu}{ds}, \quad (18)$$

which is subject to the constraint

$$p_\mu^2 = (m + V)^2. \quad (19)$$

It follows that the action can be written in Hamiltonian form as

$$S = \int d\tau (\dot{x}^\mu p_\mu - \lambda \mathcal{H}), \quad \mathcal{H} = p_\mu^2 - (m + V)^2 \quad (20)$$

where  $\lambda$  is a Lagrange multiplier enforcing the constraint (19).

The Hamilton equations read

$$\dot{x}_\mu = \{x_\mu, \lambda \mathcal{H}\} = 2\lambda p_\mu, \quad \dot{p}_\mu = \{p_\mu, \lambda \mathcal{H}\} = -2\lambda(m + V)\partial_\mu V, \quad (21)$$

from which follows that

$$\lambda = \frac{\sqrt{(\dot{x}^\rho)^2}}{2(m + V)}. \quad (22)$$

In our formalism, the potential essentially plays the role of a variable mass added to the rest mass of the particle. This is in accord with special relativity, since the external potential is fixed and does not change with the motion of the particle. So its field cannot contribute to the energy balance in any other way than effectively modifying the mass of the particle.

Again, one can choose a gauge in order to reduce to a three-dimensional problem with external time. Taking  $t = x^0$ , one has  $\{t - x_0, \lambda \mathcal{H}\} = 2\lambda p_0$ , and hence no secondary constraints arise if the potential does not depend on  $x_0$ . Moreover

$$x'_i = \frac{p_i}{p_0} = \frac{p_i}{\sqrt{p_i^2 + (m + V)^2}}, \quad p'_i = -\frac{m + V}{p_0} \partial_i V, \quad \lambda = \frac{\sqrt{1 - x_k'^2}}{2(m + V)} = \frac{1}{2p_0}. \quad (23)$$

In this case the reduced action is

$$S = \int dt \left( x'_i p_i - \sqrt{p_i^2 + (m + V)^2} \right). \quad (24)$$

The 3-dimensional Hamiltonian  $H = \sqrt{p_i^2 + (m + V)^2}$  represents the energy in the laboratory frame. It should be compared with the expression adopted in the standard formalism, namely  $H = \sqrt{p_i^2 + m^2} + V$ . This is obtained from the reduced lagrangian of the standard formalism in the gauge  $t = x_0$  and has no covariant meaning. The two Hamiltonians differ by terms of order  $(E - m)/m$ .

For  $m \gg p_i$ ,  $m \gg V$ , the expansion of both Hamiltonians gives rise to the classical expression plus higher-order corrections,

$$E \sim m + \frac{p_i^2}{2m} + V + \dots \quad (25)$$

However, with our Hamiltonian higher-order corrections are present in the expansion also for the potential term.

Like for the free particle, also in presence of a potential term, one can choose a different gauge. For example, in 1+1 dimensions, the evolution time can be made to coincide with the proper time of the particle for the choice  $t = (m + V) \left( 1 + \int^x \frac{mdx}{(m+V)^3} \right) p$ .

#### 4. Examples

We give two elementary examples that we solve in the Lagrangian formalism, and compare them with the results obtained using the standard formalism. The potentials are not Lorentz invariant and therefore the results hold only in the rest reference frame. However the equations of motion are reparametrization invariant.

##### 4.1 Harmonic oscillator

For a 1-dimensional system, the equations of motion are immediately integrated. Choosing  $t = x_0$ , the conserved energy (17) can be written

$$E = \frac{m + V}{\sqrt{1 - x'^2}}. \quad (26)$$

Easy calculations lead to the differential equation

$$x' = \frac{1}{E} \sqrt{E^2 - (m + V)^2}. \quad (27)$$

that can be immediately integrated. In the case of a harmonic oscillator,  $V = \frac{1}{2} k x^2$ , the same equation has been obtained in [1], starting from a different perspective.

The solution of (27) is given by [1]

$$x = \sqrt{\frac{E^2 - m^2}{kE}} \operatorname{sd} \left( \sqrt{\frac{k}{E}} t, \frac{E - m}{2E} \right), \quad (28)$$

with period

$$T = 4 \sqrt{\frac{E}{k}} \operatorname{K} \left( \frac{E - m}{2E} \right) \sim 2\pi \sqrt{\frac{m}{k}} \left( 1 + \frac{5}{8} \frac{E - m}{m} + \dots \right), \quad (29)$$

where sd and K are elliptic functions. Contrary to nonrelativistic mechanics, the period of the oscillations is not constant, but depends on the energy. We have written down the first term of its expansion in  $(E - m)/m$  around the nonrelativistic value.

Note that in the standard formalism, the equation of motion would have read

$$x' = \frac{\sqrt{(E - V)^2 - m^2}}{E - V} \quad (30)$$

Also this equation can be solved in terms of elliptic functions [3]. In particular, the period of the solution is

$$T \sim 2\pi\sqrt{\frac{m}{k}} \left(1 + \frac{3}{8} \frac{E-m}{m} + \dots\right) \quad (31)$$

whose dependence on energy differs from (29).

#### 4.2 Kepler problem

It is well known that the Kepler problem can be reduced to a one-dimensional problem because of the presence of two integrals of the motion related to the conservation of the angular momentum. This implies that the motion occurs on a plane and that the component of the angular momentum orthogonal to the plane is conserved. Taking  $t = x_0$  and polar coordinates  $r$  and  $\varphi$  on the plane of the motion, from (17) one has

$$E = \frac{m + V}{\sqrt{1 - r'^2 - r^2\varphi'^2}}, \quad (32)$$

while from the conservation of the norm of the angular momentum follows that

$$l = \frac{mr^2\varphi'}{\sqrt{1 - r'^2 - r^2\varphi'^2}} \quad (33)$$

is a constant. After standard computations from (32) and (33) one obtains the equation of the orbits,

$$\left(\frac{dr}{d\varphi}\right)^2 + r^2 = \frac{r^4}{l^2} [E^2 - (m + V)^2] \quad (34)$$

Defining  $u = \frac{1}{r}$  and  $V = -\frac{\alpha}{r} = -\alpha u$ , eq. (34) becomes

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{1}{l^2} [E^2 - (m - \alpha u)^2] \quad (35)$$

whose solution is given by

$$r = \frac{1}{u} = \frac{p}{1 + \varepsilon \cos q(\varphi - \varphi_0)}, \quad (36)$$

with

$$q = \sqrt{1 + \frac{\alpha^2}{l^2}}, \quad p = \frac{l^2 + \alpha^2}{\alpha m}, \quad \varepsilon = \frac{E}{m} \sqrt{1 + \frac{(E^2 - m^2) l^2}{\alpha^2 E^2}}. \quad (37)$$

These values should be compared with those obtained using the standard formalism, namely

$$q = \sqrt{1 - \frac{\alpha^2}{l^2}}, \quad p = \frac{l^2 - \alpha^2}{\alpha E}, \quad \varepsilon = \frac{m}{E} \sqrt{1 + \frac{(E^2 - m^2) l^2}{\alpha^2 m^2}}. \quad (38)$$

In the formulae above,  $\varepsilon$  is the eccentricity of the orbit, while the parameter  $p$  is related to the size of the orbit and  $q$  to the perihelion advance  $\delta = 2\pi(q^{-1} - 1)$ . The angle  $\delta$  has opposite value in our formalism with respect to the standard one. Of course, the correct value is that given by general relativity and is three times the standard one. The other parameters of the orbit calculated in our formalism differ as usual for terms of order  $(E - m)/m$  from the standard ones.

## 5. Quantization

A first quantized relativistic equation can be obtained as usual starting from the Hamiltonian (20). The Klein-Gordon equation is obtained with the substitution  $p_\mu \rightarrow i\hbar\partial_\mu$ , namely

$$\hbar^2 \left( \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x_i^2} \right) + (m + V)^2 \psi = 0. \quad (39)$$

Setting  $\psi = \phi e^{-iEt/\hbar}$ , it reduces to

$$-\hbar^2 \frac{\partial^2 \phi}{\partial x_i^2} + [(m + V)^2 - E^2] \phi = 0, \quad (40)$$

that has the form of a Schrödinger equation with  $E_{eff} = E^2 - m^2$ ,  $V_{eff} = 2mV + V^2$ . Note that if the potential  $V$  is positive definite, also  $V_{eff}$  is. This is not true in the standard formalism, leading to problems with stability.

In fact, in the standard formalism, the Klein-Gordon equation is assumed to be

$$-\hbar^2 \frac{\partial^2 \psi}{\partial x_i^2} + [m^2 - (E - V)^2] \psi = 0. \quad (41)$$

It also has the form of a Schrödinger equation with  $E_{eff} = E^2 - m^2$ ,  $V_{eff} = 2EV - V^2$ . Note that this equation cannot be derived from a classical Hamiltonian, since that is not well defined. Moreover, the presence of  $E$  in the effective potential makes it difficult to find the spectrum of the energy explicitly.

Finally, we notice that also the Dirac equation can be modified in accord with (39), as

$$[i\hbar\gamma^\mu \partial_\mu - (m + V)]\psi = 0. \quad (42)$$

## 6. Examples

We give two simple examples of solution of the Klein-Gordon equation. In the first case, only a perturbative solution is possible, while the second is constructed so that an exact solution can be found. For simplicity, in this section we set  $\hbar = 1$ .

### 6.1. Quantum harmonic oscillator

The Klein-Gordon equation (39) for a one-dimensional harmonic oscillator with potential  $V = \frac{1}{2}kx^2$  reads

$$\frac{d^2 \phi}{dx^2} = (\mu x^4 + mkx^2 - \epsilon)\phi, \quad (43)$$

where  $\epsilon = E^2 - m^2$ ,  $\mu = k^2/4$ , and has the form of a nonrelativistic biquadratic oscillator. Its energy spectrum can be obtained using standard perturbation theory, taking the quartic term as a perturbation of the nonrelativistic harmonic oscillator. The calculation has been performed in [1] and gives at first order

$$E_n = \pm \left[ m + \sqrt{\frac{k}{m}} \left( n + \frac{1}{2} + \frac{3}{4} \mu \left( n^2 + n + \frac{1}{2} \right) \right) \right], \quad (44)$$

where  $n$  is an integer.

In the standard formalism, the Klein-Gordon equation (38) reads instead

$$\frac{d^2 \phi}{dx^2} = (-\mu x^4 + Ekx^2 - \epsilon)\phi, \quad (45)$$

with  $\epsilon$  and  $\mu$  as before, and the effective potential is no longer positive definite, leading to possible instabilities [4]. Neglecting this problem, one can obtain as before the energy spectrum, which at first order in  $\hbar$ , and neglecting corrections of order  $(E - m)/m$ , yields the opposite sign for the correction, namely

$$E_n = \pm \left[ m + \sqrt{\frac{k}{m}} \left( n + \frac{1}{2} - \frac{3}{4} \mu \left( n^2 + n + \frac{1}{2} \right) \right) \right]. \quad (46)$$

## 6.2. Exactly solvable potential

An instance in which a solution of the Klein-Gordon equation (39) can be found explicitly is

$$V = m \left( \frac{1}{\cos x} - 1 \right). \quad (47)$$

This is a potential well whose value is zero at the origin and infinite at  $x = \pm \frac{\pi}{2}$ . The Klein-Gordon equation reads

$$\frac{d^2 \phi}{dx^2} - \left[ \frac{m^2}{\cos^2 x} - E^2 \right] \phi = 0. \quad (48)$$

Defining  $z = \sin x$ , eq. (48) can be put in the form

$$\frac{d^2 \phi}{dz^2} - z \frac{d\phi}{dz} - \left[ \frac{m^2}{(1-z^2)^2} - \frac{E^2}{1-z^2} \right] \phi = 0, \quad (49)$$

that in turn can be reduced to a standard hypergeometric equation with solution

$$\phi = \cos^\nu x \, F\left(\nu + E, \nu - E, \nu + \frac{1}{2}, \frac{1 + \sin x}{2}\right), \quad (50)$$

where  $F$  is a hypergeometric function and  $\nu = \frac{1}{2} (1 + \sqrt{1 + 4m^2})$ . Imposing the vanishing of  $\phi$  at  $x = \pm \frac{\pi}{2}$ , one finds the eigenfunctions

$$\phi_n = \cos^\nu x \, C_n^{(\nu)}(\sin x), \quad (51)$$

with  $C_n^{(\nu)}$  Chebyshev polynomials, and the energy spectrum

$$E = \pm \left[ m^2 + \left( n + \frac{1}{2} \right) \left( 1 + \sqrt{1 + 4m^2} \right) + n^2 \right]^{1/2}, \quad (52)$$

where  $n$  is an integer.

In the standard formalism, the Klein-Gordon equation cannot be solved analytically and therefore we do not pursue its investigation here.

## 7. Conclusions

We have shown that it is possible to formulate the problem of the motion of a particle in a positional potential in special relativity in such a way to preserve reparametrization invariance, and hence to give a consistent Hamiltonian formulation in terms of the Dirac formalism for constrained systems. Of course, potentials of the kind investigated in this paper do not occur in fundamental interactions, whose action is already invariant under reparametrization, but can be of interest for phenomenological models.

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